Eötvös Loránd University
Faculty of Science

András Sike

Applied Mathematics MSc

(andrassikeandrassike@gmail.com)

Uncertainty Principles in Harmonic Analysis

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Supervisor:

Dr. Sándor Kovács, assistant professor
Department of Numerical Analysis

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Chapter 1

Introduction

The aim of this essay is to discuss an uncertainty principle in harmonic analysis, more precisely an estimation for quaternion valued functions of one variable. Based on the idea of Stein and Shakarchi (cf. [5]) we derive an uncertainty-principle-like estimation.

Our essay is organized as follows: In the next chapter, we give a brief introduction to quaternions. Then we introduce the Schwartz space of quaternion-valued functions giving some important properties, then present a one-dimensional quaternion-valued Fourier-transform, and give some of its' properties. In the last chapter we state and prove an uncertainty principle concerning this transform.

The results whose original source is not us are labeled with a citation in brackets.

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András Sike

Chapter 2

Introduction to quaternions

Usually the word *quaternions* is attributed to W. R. Hamilton, who introduced their notion in 1843. They can be thought of as an extension of complex numbers. If $\imath \in \mathbb{C}$ denotes the complex imaginary unit the for the matrices

$$E := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I := \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix}, \qquad J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad K := \begin{bmatrix} 0 & \imath \\ \imath & 0 \end{bmatrix}$$

the following hold (cf. [3]):

$$I^2 = J^2 = K^2 = -E \qquad \text{and} \qquad IJ = -JI = K, \qquad JK = -KJ = I, \qquad KI = -IK = J.$$

It is clear that the set

$$\{\pm E, \pm I, \pm J, \pm K\}$$

is closed with respect to matrix multiplication.

The above multiplication properties of enables us to define a noncommutative field with the unit element E and null element Θ where $\Theta \in \mathbb{R}^{2\times 2}$ zero matrix.

Definition. The set

$$\mathbb{H} := \{aE + bI + cJ + dK \in \mathbb{R}^{2\times 2} : \ a,b,c,d \in \mathbb{R}\}$$

is referenced to as quaternions.

The conjugate and the modulus of a quaternion $Q = aE + bI + cJ + dK \in \mathbb{H}$ is defined by

$$\overline{Q} := Q^* \qquad \text{and} \qquad |Q| := \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Clearly,

$$QQ^* = |Q|^2 E$$

holds. For sake of brevity we rewrite the above definition in form of

$$\mathbb{H} := \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a,b,c,d \in \mathbb{R}\}, \tag{2.0.1}$$

where

$$1 := E, \qquad \mathbf{i} := I, \qquad \mathbf{j} := J, \qquad \mathbf{k} := K$$

Thus, the length and the conjugate of a quaternion

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$$

is

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2} \qquad \text{and} \qquad \overline{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$$

and clearly

$$q\overline{q} = |q|^2$$

holds.

Chapter 3

The quaternionic Schwartz space

In what follows we consider measurable quaternion-valued functions $f : \mathbb{R} \longrightarrow \mathbb{H}$. According to (2.0.1) every such a function can be expressed as

$$f(x) = f_1(x) + f_2(x)\mathbf{i} + f_3(x)\mathbf{j} + f_4(x)\mathbf{k} \qquad (x \in \mathbb{R}).$$

Introducing the quaternion-valued Schwartz space

$$\mathcal{S}(\mathbb{R},\mathbb{H}) := \left\{ f \in \mathfrak{D}^{\infty}(\mathbb{R},\mathbb{H}) : \|f\|_{k,l} := \sup_{x\in\mathbb{R}} \left| x^k f^{(\ell)}(x) \right| < \infty \quad (k,l \in \mathbb{N}_0) \right\}$$

and the quaternion-valued L^p space

$$L^p(\mathbb{R},\mathbb{H}) := \left\{ f : \mathbb{R} \to \mathbb{H} : \|f\|_p < \infty \right\},$$

where

$$\|f\|_p := \begin{cases} \|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu_1(x) \right)^{1/p} & (p \in [1, +\infty)), \\ \\ \|f\|_\infty := \operatorname{ess\,sup}(|f|) & (p = \infty) \end{cases}$$

whereupon $d\mu_1(x)$ represents the Lebesgue measure in $\mathbb{R}$; $L^\infty(\mathbb{R},\mathbb{H})$ denotes the collection of essentially bounded functions, we can prove the following statement.

Proposition. If $p \in [1,\infty]$ then $\mathcal{S}(\mathbb{R},\mathbb{H}) \subset L^p(\mathbb{R},\mathbb{H})$ holds.

Proof. Assume that $f \in \mathcal{S}(\mathbb{R},\mathbb{H})$. The every $k,l \in \mathbb{N}_0$ we have

$$\sup_{x\in\mathbb{R}} \left| x^k f^{(l)}(x) \right| < \infty,$$

in particular $(k = 0 = l)$

$$\sup_{x\in\mathbb{R}} |f(x)| < \infty$$

which means that the essential bound of f is finite:

$$\operatorname{ess\,sup}(|f|) < +\infty, \qquad \text{i.e.} \qquad f \in L^{\infty}(\mathbb{R},\mathbb{H}).$$

Using the idea in [4] we can state that if $f \in \mathcal{S}(\mathbb{R},\mathbb{H})$ then f belongs to $L^1(\mathbb{R},\mathbb{H})$, as well. Indeed, $f \in \mathcal{S}(\mathbb{R},\mathbb{H})$ implies that f is continuous (and hence measurable) and according to the definition of $\mathcal{S}(\mathbb{R},\mathbb{H})$ for sufficient $C > 0$ the estimation

$$|f(x)| \leq \frac{C}{(1+|x|)^2} \qquad (x \in \mathbb{R})$$

holds, which means that f has an integrable majorant and therefore $f \in L^1(\mathbb{R},\mathbb{H})$. If $f \in \mathcal{S}(\mathbb{R},\mathbb{H})$ and $p \in (1,+\infty)$ then with constants

$$\alpha := \|f\|_{\infty} \qquad \text{and} \qquad \beta := \sup\left\{x^2 \cdot |f(x)| \in \mathbb{R} : \ x \in \mathbb{R}\right\}$$

we have

$$\begin{aligned} \|f\|_p^p &= \int_{[-1,1]} |f(x)|^p \, d\mu_1(x) + \int_{\mathbb{R}\setminus[-1,1]} |f(x)|^p \, d\mu_1(x) \\ &\leq \alpha^p \cdot 2 + \beta^p \cdot \int_{\mathbb{R}\setminus[-1,1]} \frac{1}{|x|^{2p}} \, d\mu_1(x) < +\infty. \quad \blacksquare \end{aligned}$$

Similarly, as in [1] we can define the one-dimensional quaternion-valued Fourier transform.

Definition. The one-dimensional quaternion Fourier transform of a function $f \in \mathcal{S}(\mathbb{R},\mathbb{H})$ is defined by the formula

$$(\mathcal{F}f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}} f(x) \exp(\mathbf{i}\xi x) \, d\mu_1(x) \qquad (\xi \in \mathbb{R}). \tag{3.0.1}$$

Similarly as in [2] we can prove that for functions $f,g \in \mathcal{S}(\mathbb{R},\mathbb{H})$ the

$$\langle f,g\rangle = \frac{1}{2\pi}\langle \widehat{f},\widehat{g}\rangle$$

holds where

$$\langle f,g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)} \, d\mu_1(x).$$

From this follows the famous Plancherel's formula

$$\int_{\mathbb{R}} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{\mathbb{R}} \left|\widehat{f}(\xi)\right|^2 d\xi. \tag{3.0.2}$$

Proposition [1]. For any $f \in \mathcal{S}(\mathbb{R},\mathbb{H})$, where

$$f(x) = f_1(x) + f_2(x)\mathbf{i} + f_3(x)\mathbf{j} + f_4(x)\mathbf{k} \qquad (x \in \mathbb{R})$$

the following (conjugate symmetry) property holds:

$$\widehat{f}(\xi) = \widehat{f_1}(-\xi) - \mathbf{i}\widehat{f_2}(-\xi) - \mathbf{j}\widehat{f_3}(\xi) - \mathbf{k}\widehat{f_4}(\xi) \qquad (\xi \in \mathbb{R}).$$

Proof. The process of the proof is analogous with that of Theorem 2 in [1], we only need to use other function space, therefore we omit it. ■

The quaternion-valued Fourier transform connects a function with its' derivatives in the following manner (cf. Theorem 4 in [1]):

Theorem [1]. Let $f \in \mathcal{S}(\mathbb{R},\mathbb{H})$. Then the equality

$$\widehat{f^{(n)}}(\xi) = \hat{f}(\xi) \cdot \xi^n \cdot (-\mathbf{i})^n$$

holds for any $n \in \mathbb{N}_0$ number, where $f^{(n)}$ denotes the n-th derivative of f.

Chapter 4

Uncertainty principle for the one-dimenstional quaternionic Fourier-transform

Proposition. For any number $t \in \mathbb{R}$, the operators defined by

$$(A_t f)(x) := f'(x) + txf(x), \qquad \text{and} \qquad (A_t^* f)(x) := -f'(x) + txf(x)$$

are Hermitian adjoints, that is,

$$\langle A_t f, g\rangle = \langle f, A_t^* g\rangle \qquad (f, g \in \mathcal{S}(\mathbb{R},\mathbb{H})).$$

Furthermore, for any $f \in \mathcal{S}(\mathbb{R},\mathbb{H})$ the following holds:

$$\langle A_t^* A_t f, f\rangle \geq 0.$$

Proof.

Step 1. After integration by parts, the left side is equal to $\langle A_t f, g\rangle$

$$= \int_{\mathbb{R}} (f'(x) + txf(x))\overline{g(x)}\, d\mu_1(x) = \int_{\mathbb{R}} f'(x)\overline{g(x)}\, d\mu_1(x) + \int_{\mathbb{R}} txf(x)\overline{g(x)}\, d\mu_1(x)$$

$$= 0 - \int_{\mathbb{R}} f(x)\overline{g'(x)}\, d\mu_1(x) + \int_{\mathbb{R}} txf(x)\overline{g(x)}\, d\mu_1(x),$$

and we can rewrite the right side to the form $\langle f, A_t^* g\rangle$

$$= \int_{\mathbb{R}} f(x)\overline{(-g'(x) + txg(x))}\, d\mu_1(x) = \int_{\mathbb{R}} -f(x)\overline{g'(x)}\, d\mu_1(x) + \int_{\mathbb{R}} f(x)\overline{txg(x)}\, d\mu_1(x)$$

$$= -\int_{\mathbb{R}} f(x)\overline{g'(x)}\, d\mu_1(x) + \int_{\mathbb{R}} txf(x)\overline{g(x)}\, d\mu_1(x).$$

This means that

$$\langle A_t f,g\rangle = -\int_{\mathbb{R}} f(x)\overline{g'(x)}\,d\mu_1(x) + \int_{\mathbb{R}} txf(x)\overline{g(x)}\,d\mu_1(x) = \langle f,A_t^* g\rangle.$$

Step 2. By using the adjoint property of the two operators and the nonnegativity of the norm, we get that

$$\langle A_t^* A_t f,f\rangle = \langle A_t f,A_t f\rangle = \|A_t f\|_{L_2}^2 \geq 0. \qquad \blacksquare$$

Remark. Note that we can rewrite the operator $A_t^* A_t$ in the following manner:

$$\begin{aligned}(A_t^* A_t f)(x) &= A_t^*(f'(x) + txf(x)) = -f''(x) - (txf(x))' + txf'(x) + tx \cdot tx \cdot f(x) = \\ &= -f''(x) + t^2x^2f(x) - txf'(x) - t \cdot 1 \cdot f(x) + txf'(x) = \\ &= t^2x^2f(x) - tf(x) - f''(x) \quad (x \in \mathbb{R}).\end{aligned}$$

Now, we can use the method outlined in Problem 23 formulated in [5].

Proposition. Suppose that the function $f \in \mathcal{S}(\mathbb{R},\mathbb{H})$ is normed, that is $\|f\|^2 := \langle f,f\rangle = 1$ holds. Then

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2 \, d\mu_1(x)\right) \cdot \left(\int_{\mathbb{R}} |f'(x)|^2\right) d\mu_1(x) \geq \frac{1}{4}.$$

Proof. We start by substituting for and rewriting the inequality from the previous proposition.

$$0 \leq \langle A_t^* A_t f; f\rangle = \int_{\mathbb{R}} t^2x^2f(x)\overline{f(x)}\,d\mu_1(x) - \int_{\mathbb{R}} tf(x)\overline{f(x)}\,d\mu_1(x) - \int_{\mathbb{R}} f''(x)\overline{f(x)}\,d\mu_1(x).$$

We know that for any quaternion multiplied by its' conjugate is its' length squared and therefore

$$\begin{aligned}0 &\leq \langle A_t^* A_t f; f\rangle \\ &= \int_{\mathbb{R}} t^2x^2f(x)\overline{f(x)}\,d\mu_1(x) - \int_{\mathbb{R}} tf(x)\overline{f(x)}\,d\mu_1(x) - \int_{\mathbb{R}} f''(x)\overline{f(x)}\,d\mu_1(x) \\ &= t^2 \cdot \int_{\mathbb{R}} x^2f(x)\overline{f(x)}\,d\mu_1(x) - t \cdot \langle f,f\rangle - \int_{\mathbb{R}} f''(x)\overline{f(x)}\,d\mu_1(x) \\ &= t^2 \cdot \int_{\mathbb{R}} x^2f(x)\overline{f(x)}\,d\mu_1(x) - t \cdot 1 - \int_{\mathbb{R}} f''(x)\overline{f(x)}\,d\mu_1(x)\end{aligned}$$

and we have obtained a quadratic polynomial in t, whose principal coefficient is nonnegative (since $x^2|f(x)|^2$ is nonnegative). This can only be nonnegative at every point if its' determinant is nonpositive, that is,

$$1 - 4\left(\int_{\mathbb{R}} x^2 |f(x)|^2 \, d\mu_1(x)\right) \cdot \left(\int_{\mathbb{R}} |f'(x)|^2 \, d\mu_1(x)\right) \leq 0,$$

and by rewriting this we obtain the inequality stated in the theorem:

$$\left(\int_{\mathbb{R}} x^2 \left|f(x)\right|^2 \, d\mu_1(x)\right) \cdot \left(\int_{\mathbb{R}} \left|f'(x)\right|^2 \, d\mu_1(x)\right) \geq \frac{1}{4}. \qquad \blacksquare$$

Remark. Using Plancherel's formula (3.0.2) and the identity that connects a function's the quaternionic Fourier transform to its' derivative, we can write the above inequality as

$$\begin{aligned} \frac{1}{4} \;&\leq\; \left(\int_{\mathbb{R}} x^2 \left|f(x)\right|^2 \, d\mu_1(x)\right) \cdot \left(\int_{\mathbb{R}} \left|f'(x)\right|^2 \, d\mu_1(x)\right) \\ &=\; \left(\int_{\mathbb{R}} x^2 \left|f(x)\right|^2 \, d\mu_1(x)\right) \cdot \left(\frac{1}{2\pi} \int_{\mathbb{R}} \left|\widehat{f'}(\xi)\right|^2 \, d\mu_1(\xi)\right) = \qquad , \\ &=\; \left(\int_{\mathbb{R}} x^2 \left|f(x)\right|^2 \, d\mu_1(x)\right) \cdot \left(\frac{1}{2\pi} \int_{\mathbb{R}} \left|\widehat{f}(\xi) \cdot \xi \cdot (-\mathbf{i})\right|^2 \, d\mu_1(\xi)\right) \end{aligned}$$

and so we can state our main result which is the following uncertainty principle:

Proposition. Suppose that $f \in \mathcal{S}(\mathbb{R},\mathbb{H})$ and $\|f\|^2 = 1$. Then

$$\left(\int_{\mathbb{R}} x^2 \left|f(x)\right|^2 \, d\mu_1(x)\right) \cdot \left(\int_{\mathbb{R}} \left|\widehat{f}(\xi) \cdot \xi \cdot (-\mathbf{i})\right|^2 \, d\mu_1(\xi)\right) \geq \frac{\pi}{2}.$$

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